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The Stationary Nonlinear Boltzmann Equation in a Couette Setting with Multiple, Isolated L^q-solutions and Hydrodynamic Limits

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This paper studies the stationary nonlinear Boltzmann equation for hard forces, in a Couette setting between two coaxial, rotating cylinders with given indata of Maxwellian type on the cylinders. *A priori* estimates are obtained mainly in L^2, leading to multiple, isolated solutions together with a hydrodynamic limit control, based on asymptotic expansions together with a rest term.

KEY WORDS: Asymptotic techniques; Couette setting; hydrodynamic limit; isolated solutions; multiple solutions; stationary nonlinear Boltzmann equation.

1. INTRODUCTION

The asymptotic kinetic approach in a sharp mathematical form has its roots in works by Grad and Kogan in the 1960s (see refs. 15, 16, 21 and references therein). A number of important results followed, concerning the nonlinear stationary Boltzmann equation in $\mathbb{R}^n$ in the close to equilibrium case (refs. 17–19, 29 and others), where techniques of a general scope were used, such as contraction mappings (see also ref. 13). Stationary problems in small domains were solved in a similar way (e.g. refs. 20 and 24), and the unique solvability of internal, stationary problems for the Boltzmann equation at large Knudsen numbers was likewise established (cf. ref. 22). In ref. 7, a kinetic description of a gas between two plates at different temperatures and no mass flux was studied in the case of a small mean free path for the nonlinear stationary Boltzmann equation under diffuse reflection boundary conditions. Stationary, fully nonlinear hydrodynamic limits, were treated in the papers.[11,12]

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Solutions to half-space problems for the Boltzmann equation play an important role as boundary layers for hydrodynamic limits of such solutions to the Boltzmann equation, when the mean free path tends to zero. This has been extensively studied in the linear context, using functional analytic and energy methods (refs. 8, 14 and others). Also for the nonlinear case, related results have been obtained recently.[30]

A wide range of questions of the above types have been addressed in a perspective of asymptotic analysis and numerical studies for the BGK and Boltzmann equations by Sone and his group (see the monograph[25] and its extensive references).

With a loss of the uniqueness aspects, further away from equilibrium weak compactness arguments may be employed instead of the earlier contraction mappings to prove existence, and in the stationary case usually involving entropy dissipation control for the sharpest results. That is the case in the spatially n-dimensional Povzner and one space-dimensional Boltzmann papers[2–4], where stationary solutions are obtained via weak L^1-compactness under no other restrictions than Grad's angular cut-off. The basic compactness argument used in those cases, is not fully available for the Boltzmann equation itself in more than one space dimension. However, in the spatially n-dimensional case the entropy dissipation estimate still allows different but weaker control mechanisms, which also lead to existence results (see ref. 5). There, in contrast to the cases mentioned before, complete results are so far only obtained when the velocities smaller in norm than some $\eta > 0$, are suppressed.

The present study is set in the close to equilibrium frame and gives a mathematically rigorous study of the stationary nonlinear Boltzmann equation between two coaxial cylinders A and B, with Maxwellian ingoing boundary values, and includes small mean free path asymptotics. This two-roll problem is extensively treated from a numerical and asymptotic perspective in ref. 26, to which we also refer for a more complete discussion of the applied aspects. See ref. 6 for an existence study (but with no control of uniqueness or local uniqueness) by weak compactness in the case of more general two-roll problems also far from equilibrium and with no suppression of small velocities.

The boundary values and the solutions are assumed to be axially and circumferentially uniform in the space variables. Then, with (r, θ, z) and (v_r, v_θ, v_z) respectively denoting the spatial cylindrical coordinates and the corresponding velocity coordinates, a distribution function may be written as $f = f(r, v_r, v_\theta, v_z)$, and the Boltzmann equation becomes

$$v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f = \frac{1}{\epsilon^m} Q(f, f), \quad r \in (r_A, r_B), \quad (v_r, v_\theta, v_z) \in \mathbb{R}^3. \tag{1.1}$$

The Maxwellian ingoing boundary data under study are

$$\begin{aligned} \gamma^+ f(r_A, v) &= (2\pi)^{-\frac{3}{2}} e^{\frac{1}{2}(-v_r^2-(v_\theta-\epsilon u_{\theta A1})^2-v_z^2)}, \quad v_r > 0, \\ \gamma^+ f(r_B, v) &= (2\pi)^{-\frac{3}{2}} \frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}} e^{\frac{1}{2}(-\frac{1}{1+\tau_B}(v_r^2+(v_\theta-\epsilon u_{\theta B1})^2+v_z^2))}, \quad v_r < 0. \end{aligned} \tag{1.2}$$

Here

$$\begin{aligned} Nf &:= v_\theta^2 \frac{\partial f}{\partial v_r} - v_\theta v_r \frac{\partial f}{\partial v_\theta}, \\ Q(f,f)(v) &:= \int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)(f(v')f(v'_*)-f(v)f(v_*))dv_* d\omega. \end{aligned} \tag{1.3}$$

The kernel $B = |v-v_*|^\beta b(\theta)$, $b \in L^1_+(S^2)$, $0 \leqslant \beta \leqslant 1$, is of hard force type (ref. 9) and assumed to belong to the Grad class, that is with its terms suitably majorized by the corresponding ones for the hard sphere model (cf. ref. 23). The case $\beta = 0$ corresponds to Maxwellian molecules, and $\beta = 1$ to hard spheres. For a bifurcation case also included in this paper, but not for the main results about isolated existence and strict positivity per se, the kernel is assumed to imply (2.26) below. That condition is discussed in the text directly following (2.26). In the present setup, it is enough to consider functions which are even in the axial velocity variable v_z. Take the radii as $r_A = 1$, $r_B > 1$, and let ϵ^m denote the Knudsen number. The given rotational velocities of the inner and outer cylinders are $u_{\theta A} = \epsilon u_{\theta A1}$ and $u_{\theta B} = \epsilon u_{\theta B1}$, respectively. The parameter ϵ measures the depth of the (suction) boundary layer. The nondimensional density, perturbed temperature and saturated pressure are

$$\omega_B = \frac{\epsilon^2}{1+\tau_B}(P_{SB2} - \tau_{B2}), \quad \tau_B = \epsilon^2 \tau_{B2}, \quad P_{SB} = \epsilon^2 P_{SB2},$$

where lower indices A, B refer to the boundary points, and lower indices $1, 2, \ldots$ refer to the order in ϵ. In the bifurcation case, an extra coupling is added between boundary pressure and velocity,

$$P_{SB2} - \frac{r_B^2 - 1}{r_B^2} u_{\theta A1}^2 = \Delta\epsilon,$$

or

$$\omega_B = \frac{\epsilon^2}{1+\epsilon^2\tau_{B2}} \left(\frac{r_B^2-1}{r_B^2} u_{\theta A1}^2 - \tau_{B2} + \Delta\epsilon \right). \tag{1.4}$$

For $1 \leqslant q \leqslant +\infty$, denote by $\| . \|_q$ the usual Lebesgue norm in L^q, and set

$$\tilde{L}^q := \left\{ f; \mid f \mid_q := \left(\int M(v) \left(\int \mid f(x,v) \mid^q dx \right)^{\frac{2}{q}} dv \right)^{\frac{1}{2}} < +\infty \right\},$$

where $M = (2\pi)^{-\frac{3}{2}} \exp(-\frac{v^2}{2})$. In order to fix the asymptotic expansions, this paper focuses on the case $m = 4$ in (1.1), i.e. takes the Knudsen number as ϵ^4. The central result can then be stated as follows.

Theorem 1.1. Assume that $u_{\theta A1}$, $u_{\theta B1}$ and τ_{B2} are small enough, and that $(r_B^2 - 1)\left(\frac{u_{\theta A1}}{r_B}\right)^2 > P_{SB2} > (r_B^2 - 1)u_{\theta B1}^2$, where $1 + P_{SB2}$ is the non-dimensional saturated pressure at r_B. For the quantity ϵ positive and small enough, there is a solution f_ϵ^+ of (1.1–2) isolated in L^1 with positive leading order radial velocity ϵ^2, and another f_ϵ^- with negative leading order radial velocity ϵ^2. They satisfy $M^{-1} f_\epsilon^+, \quad M^{-1} f_\epsilon^- \in \tilde{L}^\infty$,

$$\int M^{-1} \text{supess}_{r \in (r_A, r_B)} \mid f_\epsilon^j (r,v) \mid^2 dv < +\infty, \quad j = \pm.$$

There is also a similar third isolated solution with leading order radial velocity of order ϵ^4. The hydrodynamic moments of these three solutions converge to solutions of the corresponding limiting fluid equations at leading order, when $\epsilon \to 0$.

Under the additional bifurcation coupling (1.4), the corresponding result becomes

Theorem 1.2. Assume that $u_{\theta A1}$, $u_{\theta B1}$ and τ_{B2} are small enough and that $(u_{\theta B1} r_B - u_{\theta A1})(3u_{\theta A1} + u_{\theta B1} r_B)(A + 5D) > 0$, where A, D are defined in (2.9) below. There is a negative value Δ_{bif} of the parameter Δ, such that for the quantity $\Delta_{bif} - \Delta$ positive and small enough, there are for the quantity ϵ positive and small enough, two isolated, non-negative L^1-solutions f_ϵ^j, $j = 1, 2$ of (1.1–2) coexisting with $M^{-1} f_\epsilon^j \in \tilde{L}^\infty$,

$$\int M^{-1} \text{supess}_{r \in (r_A, r_B)} \mid f_\epsilon^j (r,v) \mid^2 dv < +\infty.$$

The two solutions have different outward radial bulk velocities of order ϵ^3. For fixed ϵ, they converge to the same solution, when Δ increases to Δ_{bif}. Their hydrodynamic moments converge to solutions of the corresponding limiting fluid equations at leading order, when $\epsilon \to 0$.

Remark. The approach of the paper has wider applicability; for instance, analogous results hold for $m \geqslant 3$ in Theorem 1.1 and $m \geqslant 4$ in Theorem 1.2, and for all cases of the two-roll problem treated in ref. 26. We expect the techniques developed here, also to be useful in the study of related problems, such as the Taylor–Couette frame of ref. 27, the Bénard asymptotics of ref. 28, and the two-component gases of ref. 1. In particular a paper on the Taylor–Couette case is under preparation where we also present an approach to strict positivity for this type of solutions.

Write $R = f_{\text{rest}} = P_0 f_{\text{rest}} + (I - P_0) f_{\text{rest}} = R_{\|} + R_{\perp}$, where P_0 is the orthogonal projection operator onto the hydrodynamic part $P_0 f_{\text{rest}}$, and

$$f = M(1 + \varphi + \epsilon^{j_0} f_{\text{rest}}) \text{ with } \varphi = \sum_{1}^{j_1} \epsilon^j \varphi^j. \tag{1.5}$$

Here $\sum_1^{j_1} \epsilon^j \varphi^j$ is an asymptotic expansion with the boundary value of the φ^j-terms up to some suitable order $\leqslant j_1$ equal to the terms of corresponding order in the ϵ-expansions of (1.2), and based on a splitting into interior Hilbert expansion and boundary layers. A central part of the paper is devoted to a rigorous study of the rest term $R = f_{\text{rest}}$ in $\tilde{L}^q$, using as ingoing boundary values what remains of (1.2) after correction for the asymptotic expansion. The rest term problem is solved by a contraction mapping iteration.

The problem area, the plan of the paper, and the main results are introduced in the present Section 1.

Section 2 is devoted to the asymptotic expansion, adapting the presentation in ref. 26 to the needs of this paper. For the convenience of the reader, the description is fairly self-contained and includes details of particular relevance. Section 3 discusses some *a priori* estimates for the rest term. An estimate for the nonhydrodynamic part in $\tilde{L}^2$ is obtained from Green's formula. The study of the hydrodynamic part in $\tilde{L}^2$ utilizes the couplings between certain moments, and involves details about the terms in the asymptotic expansion, for the hydrodynamic ones up to order ϵ^{j_0}. A new type of preliminary rearrangements of the equation is introduced to increase the ϵ-order of certain nonhydrodynamic terms and to remove the influence of otherwise troublesome hydrodynamic terms. This is a step with origin in the fact that here the boundary scalings (of order ϵ) are larger than the Knudsen number (ϵ^4).

Section 4 deals with the existence problem for the rest term via a contraction mapping construction, which uses the *a priori* estimates of Section 3.

2. THE ASYMPTOTIC EXPANSION

The asymptotic expansion is here carried out in the frame of Theorem 1.2 in order to include the aspects which are important for the paper. Write the solution of (1.1–2) as $f = M(1+\Phi)$. Then the new unknown $\Phi(r, v_r, v_\theta)$ should be solution to

$$v_r \frac{\partial \Phi}{\partial r} + \frac{1}{r} N\Phi = \frac{1}{\epsilon^4}(\tilde{L}\Phi + \tilde{J}(\Phi, \Phi)), \tag{2.1}$$

$$\Phi(1, v) = e^{\frac{1}{2}(v_\theta^2 - (v_\theta - \epsilon u_{\theta A1})^2)} - 1, \quad v_r > 0, \tag{2.2}$$

$$\Phi(r_B, v) = \frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}} e^{\frac{1}{2}(v^2 - \frac{1}{1+\tau_B}(v_r^2 + (v_\theta - \epsilon u_{\theta B1})^2 + v_z^2))} - 1, \quad v_r < 0. \tag{2.3}$$

Here $\tilde{J}$ is the rescaled quadratic Boltzmann collision operator,

$$\begin{aligned}\tilde{J}(\Phi, \psi)(v) := \frac{1}{2}\int_{\mathbb{R}^3 \times S^2} & B(v - v_*, \omega) M(v_*)(\Phi(v')\psi(v'_*) + \Phi(v'_*)\psi(v') \\ & -\Phi(v_*)\psi(v) - \Phi(v)\psi(v_*)) dv_* d\omega,\end{aligned}$$

and $\tilde{L}$ is this operator linearized around 1,

$$\begin{aligned}(\tilde{L}\Phi)(v) := \int_{\mathbb{R}^3 \times S^2} & B(v - v_*, \omega) M(v_*)(\Phi(v') + \Phi(v'_*) - \Phi(v_*) \\ & -\Phi(v)) dv_* d\omega = \tilde{K}(\Phi) - \tilde{\nu}\Phi.\end{aligned}$$

Denote by $(\Phi_{Ai})_{1 \leqslant i \leqslant j}$ resp. $(\Phi_{Bi})_{1 \leqslant i \leqslant j}$, the first to j-th order terms of $\Phi(r_A)$ resp. $\Phi(r_B)$, with respect to ϵ. E.g. for $j = 4$

$$\begin{aligned}\sum_i^4 \epsilon^i \Phi_{Ai}(v) = {} & \epsilon u_{\theta A1} v_\theta + \epsilon^2 \frac{u_{\theta A1}^2}{2}(-1 + v_\theta^2) \\ & + \epsilon^3 \frac{u_{\theta A1}^3}{2}\left(-v_\theta + \frac{1}{3}v_\theta^3\right) + \epsilon^4 \frac{u_{\theta A1}^4}{4}\left(\frac{1}{2} - v_\theta^2 + \frac{1}{6}v_\theta^4\right), \quad v_r > 0\end{aligned}$$

$$
\begin{aligned}
\sum_1^4 \epsilon^i \Phi_{Bi}(v) = {} & \epsilon u_{\theta B1} v_\theta + \epsilon^2 \left(\frac{r_B^2-1}{2r_B^2} u_{\theta A1}^2 - \frac{5}{2}\tau_{B2} - \frac{1}{2}u_{\theta B1}^2 + \frac{1}{2}u_{\theta B1}^2 v_\theta^2 + \frac{1}{2}\tau_{B2} v^2 \right) \\
& + \epsilon^3 \left(\Delta + u_{\theta B1} v_\theta \left(\frac{r_B^2-1}{r_B^2} u_{\theta A1}^2 - \frac{7}{2}\tau_{B2} - u_{\theta B1}^2 + \frac{1}{3} u_{\theta B1}^2 v_\theta^2 + \frac{1}{2}\tau_{B2} v^2 \right) \right) \\
& + \epsilon^4 \left(\frac{7}{4} u_{\theta B1}^2 \tau_{B2} + \frac{1}{8} u_{\theta B1}^4 + \frac{27}{8}\tau_{B2}^2 - \frac{r_B^2-1}{4r_B^2} u_{\theta A1}^2 u_{\theta B1}^2 - \frac{3}{4}\frac{r_B^2-1}{r_B^2} u_{\theta A1}^2 \tau_{B2} + \Delta u_{\theta B1} v_\theta \right. \\
& + \frac{1}{4} u_{\theta B1}^2 \left(\frac{r_B^2-1}{r_B^2} u_{\theta A1}^2 - 7\tau_{B2} \right) v_\theta^2 + \frac{1}{4}\tau_{B2} \left(\frac{r_B^2-1}{r_B^2} u_{\theta A1}^2 - 7\tau_{B2} - u_{\theta B1}^2 \right) v^2 \\
& \left. + \frac{1}{8}\tau_{B2}^2 v^4 + \frac{1}{24} u_{\theta B1}^4 v_\theta^4 \right), \quad v_r < 0.
\end{aligned}
$$

A solution Φ will be determined as an approximate solution φ of order j_1 with boundary values of order i being Φ_{Ai} resp. Φ_{Bi} for $1 \leqslant i \leqslant j_0$, plus a rest term $R = f_{\text{rest}}$,

$$\Phi(r,v) = \varphi(r,v) + \epsilon^{j_0} R(r,v).$$

We shall here give a fairly detailed discussion of the asymptotic expansion for $j_0 = 4$, $j_1 = 4$. Similar expansions hold for other values of $j_1 \geqslant j_0$ (cf ref. 25), and such variants will be used in later sections. For $j_1 = 4$,

$$
\begin{aligned}
\varphi(r,v) = {} & \epsilon \left(\Phi_{H1}(r,v) + \Phi_{W1}\left(\frac{r-r_B}{\epsilon}, v\right)\right) + \epsilon^2 \left(\Phi_{H2}(r,v) + \Phi_{W2}\left(\frac{r-r_B}{\epsilon}, v\right)\right) \\
& + \epsilon^3 \left(\Phi_{H3}(r,v) + \Phi_{W3}\left(\frac{r-r_B}{\epsilon}, v\right) + \Phi_{K3A}\left(\frac{r-1}{\epsilon^4}, v\right) + \Phi_{K3B}\left(\frac{r-r_B}{\epsilon^4}, v\right)\right) \\
& + \epsilon^4 \left(\Phi_{H4}(r,v) + \Phi_{W4}\left(\frac{r-r_B}{\epsilon}, v\right) + \Phi_{K4A}\left(\frac{r-1}{\epsilon^4}, v\right) + \Phi_{K4B}\left(\frac{r-r_B}{\epsilon^4}, v\right)\right),
\end{aligned}
\tag{2.4}
$$

with

$$
\begin{aligned}
\int \Phi_{H1}(.,v)(1, v_r, v^2) M(v)\,dv &= \int \Phi_{W1}(.,v)(1, v_r, v^2) M(v)\,dv \\
&= \int \Phi_{H2}(.,v) v_r M(v)\,dv = 0,
\end{aligned}
\tag{2.5}
$$

$$\lim_{\frac{r-r_B}{\epsilon} \to -\infty} \Phi_{Wi}\left(\frac{r-r_B}{\epsilon}, v\right) = 0, \quad 1 \leqslant i \leqslant 4, \tag{2.6}$$

$$\lim_{\frac{r-1}{\epsilon^4} \to +\infty} \Phi_{KiA}\left(\frac{r-1}{\epsilon^4}, v\right) = 0, \qquad \lim_{\frac{r-r_B}{\epsilon^4} \to -\infty} \Phi_{KiB}\left(\frac{r-r_B}{\epsilon^4}, v\right) = 0, \quad 3 \leqslant i \leqslant 4. \tag{2.7}$$

Here $(\epsilon\Phi_{H1}+\epsilon^2\Phi_{H2}+\epsilon^3\Phi_{H3}+\epsilon^4\Phi_{H4})(r,v)$ denotes the truncation up to fourth order of a formal expansion $\sum_{k\geqslant 1}\epsilon^k\Phi_{Hk}(r,v)$. The sum $(\epsilon\Phi_{W1}+\epsilon^2\Phi_{W2})(\frac{r-r_B}{\epsilon},v)$ consists of correction terms allowing the boundary conditions to be satisfied to first and second order. They correspond to a suction boundary layer at r_B. Supplementary boundary layers of Knudsen type, described by

$$\epsilon^3\left(\Phi_{K3A}\left(\frac{r-1}{\epsilon^4},v\right)+\Phi_{K3B}\left(\frac{r-r_B}{\epsilon^4},v\right)\right)+\epsilon^4\left(\Phi_{K4A}\left(\frac{r-1}{\epsilon^4},v\right)\right.$$
$$\left.+\,\Phi_{K4B}\left(\frac{r-r_B}{\epsilon^4},v\right)\right),$$

are required in order to have the boundary conditions satisfied at third and fourth orders.

Uniqueness statements are given modulo possible shifts of terms between the asymptotic expansion from fourth order on, and the rest term. Recall (see ref. 10) that $\tilde{L}(v_\theta v_r\bar{B})=v_\theta v_r$, $\tilde{L}(v_r\bar{A})=v_r(v^2-5)$ for some functions $\bar{B}(|v|)$ and $\bar{A}(|v|)$, with $v_\theta v_r\bar{B}(|v|)$ and $v_r\bar{A}(|v|)$ bounded in the $(,)_M$-norm. Set $w_1:=\int v_r^2v_\theta^2\bar{B}Mdv$, and let $g(\eta,v)$ be the solution to the half-space problem

$$\begin{aligned} v_r\frac{\partial g}{\partial\eta}&=\tilde{L}g,\quad \eta>0,\ \ v\in\mathbb{R}^3,\\ g(0,v)&=0,\quad v_r>0,\\ \int g(\eta,v)v_rM(v)dv&=1,\quad a.a.\eta>0. \end{aligned}\tag{2.8}$$

From the approaches in refs. 8 and 14 including their point-wise estimates, it follows that there are constants A, D, and E, such that with respect to $\tilde{L}^q$

$$\lim_{\eta\to+\infty}g(\eta,v)=A+Dv^2+Ev_\theta+v_r.\tag{2.9}$$

Proposition 2.1. Assume that

$$(u_{\theta B1}r_B-u_{\theta A1})(u_{\theta B1}r_B+3u_{\theta A1})(A+5D)>0,$$

and set

$$\Delta_{bif}:=-\left(2w_1\frac{r_B+1}{r_B^3}(A+5D)(r_Bu_{\theta B1}-u_{\theta A1})\,(r_Bu_{\theta B1}+3u_{\theta A1})\right)^{\frac{1}{2}}.$$

For $\Delta > \Delta_{bif}$, there is no solution ψ in the family defined in (2.4–7).

For $\Delta = \Delta_{bif}$, there is a unique solution ψ in the family defined in (2.4–7).

For $\Delta < \Delta_{bif}$, there are two solutions ψ in the family defined in (2.4–7).

Proof of Proposition 2.1. Define $Y := \frac{r-r_B}{\epsilon}$, and let the expansions $\sum_{k\geqslant 1}\epsilon^k \Phi_{Hk}(r,v)$ and $\sum_{k\geqslant 1}\epsilon^k(\Phi_{Hk}(r_B,v)+\Phi_{Wk}(\frac{r-r_B}{\epsilon},v))$ formally satisfy (2.1). Then,

$$\begin{aligned}\tilde{L}\Phi_{H1} &= \tilde{L}\Phi_{H2}+\tilde{J}(\Phi_{H1},\Phi_{H1})=\tilde{L}\Phi_{H3}+2\tilde{J}(\Phi_{H1},\Phi_{H2})\\ &= \tilde{L}\Phi_{H4}+2\tilde{J}(\Phi_{H1},\Phi_{H3})+\tilde{J}(\Phi_{H2},\Phi_{H2})=0,\end{aligned} \tag{2.10}$$

$$v_r\frac{\partial\Phi_{Hk-4}}{\partial r}+\frac{1}{r}N\Phi_{Hk-4} = \tilde{L}\Phi_{Hk}+\sum_{j=1}^{k-1}\tilde{J}(\Phi_{Hj},\Phi_{Hk-j}),\quad k\geqslant 5, \tag{2.11}$$

and

$$\begin{aligned}\tilde{L}\Phi_{W1} &= \tilde{L}\Phi_{W2}+\tilde{J}(\Phi_{W1},2\Phi_{H1}(r_B,.)+\Phi_{W1})\\ &= \tilde{L}\Phi_{W3}+2\tilde{J}(\Phi_{H1}(r_B,.)+\Phi_{W1},\Phi_{W2})+2\tilde{J}(\Phi_{W1},\Phi_{H2}(r_B,.)+Y\Phi'_{H1}(r_B,.))\\ &= \tilde{L}\Phi_{W4}+2\tilde{J}(\Phi_{W3},\Phi_{H1}(r_B,.)+\Phi_{W1})+\tilde{J}(\Phi_{W2},\Phi_{W2}+2\Phi_{H2}(r_B,.)\\ &\quad +2Y\Phi'_{H1}(r_B,.))+2\tilde{J}(\Phi_{W1},\Phi_{H3}(r_B,.)+Y\Phi'_{H2}(r_B,.)\\ &\quad +\frac{Y^2}{2}\Phi''_{H1}(r_B,.))-v_r\frac{\partial\Phi_{W1}}{\partial Y}=0,\end{aligned} \tag{2.12}$$

$$\begin{aligned}&v_r\frac{\partial\Phi_{Wk-3}}{\partial Y}+\frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i\left(\frac{Y}{r_B}\right)^i N(\Phi_{Hk-4-i}(r_B,.)+\Phi_{Wk-4-i})\\ &\quad =\tilde{L}\Phi_{Wk}+\sum_{j=1}^{k-1}\tilde{J}(2\Phi_{Hj}(r_B,.)+\Phi_{Wj},\Phi_{Wk-j}),\quad k\geqslant 5.\end{aligned} \tag{2.13}$$

By (2.5) and (2.10), $\Phi_{H1}(r,v)=b_1(r)v_\theta$ for some function b_1, and $\Phi_{Hi}, i\geqslant 2$ split into a hydrodynamical part $a_i(r)+d_i(r)v^2+b_i(r)v_\theta+c_i(r)v_r$ and a non-hydrodynamic part involving Hilbert terms of lower order than i. In particular for $1\leqslant j\leqslant 4$ we get

$$\begin{aligned}\Phi_{H1}(r,v) &= b_1(r)v_\theta,\\ \Phi_{H2} &= a_2+d_2v^2+b_2v_\theta+\frac{1}{2}b_1^2v_\theta^2,\\ \Phi_{H3} &= a_3+d_3v^2+b_3v_\theta+c_3v_r+b_1d_2v_\theta v^2+b_1b_2v_\theta^2+\frac{1}{6}b_1^3v_\theta^3,\end{aligned}$$

$$\Phi_{H4} = a_4 + d_4 v^2 + b_4 v_\theta + c_4 v_r + (b_1 d_3 + b_2 d_2) v_\theta v^2 + \left(b_1 b_3 + \frac{1}{2} b_2^2 - \frac{1}{2} b_1^2 a_2\right) v_\theta^2 + b_1 c_3 v_r v_\theta + \frac{1}{2} b_1^2 b_2 v_\theta^3 + \frac{1}{2} d_2^2 v^4 + \frac{1}{24} b_1^4 v_\theta^4 + \frac{1}{2} b_1^2 d_2 v_\theta^2 v^2.$$

Equations (2.11) have solutions if and only if the following compatibility conditions hold,

$$\int \left(v_r \frac{\partial \Phi_{Hi}}{\partial r} + \frac{1}{r} N \Phi_{Hi}\right)(1, v^2 - 5, v_\theta, v_r) M(v) dv = 0, \quad i \geqslant 1.$$

They provide first-order differential equations for the functions $a_i(r)$, $b_i(r)$, $c_i(r)$ and $d_i(r)$, $i \geqslant 1$. In particular,

$$(r b_1)' = 0, \quad (10 d_2 + b_1^2)' = 0, \tag{2.14}$$

$$(r^2 c_3 b_2)' = w_1 r^2 \left(b_1' - \frac{1}{r} b_1\right)' + (2 w_1 - w_2) r \left(b_1' - \frac{1}{r} b_1\right),$$

$$\left(a_2 + 5 d_2 + \frac{1}{2} b_1^2\right)' = \frac{1}{r} b_1^2, \tag{2.15}$$

$$(a_3 + 5 d_3 + b_1 b_2)' = \frac{2}{r} b_1 b_2, \tag{2.16}$$

$$(r c_3)' = 0, \tag{2.17}$$

$$\left(a_4 + 5 d_4 + b_1 b_3 + \frac{1}{2} b_2^2 - \frac{1}{2} b_1^2 a_2 + \frac{35}{2} d_2^2 + \frac{7}{2} b_1^2 d_2\right)' = \frac{2}{r}\left(b_1 b_3 + \frac{1}{2} b_2^2 - \frac{1}{2} b_1^2 a_2\right) + \frac{1}{2r} b_1^4 + \frac{7}{r} b_1^2 d_2, \tag{2.18}$$

$$(r c_4)' = 0.$$

Together with the boundary condition (2.2) at first and second orders, this fixes

$$\Phi_{H1}(r, v) = \frac{u_{\theta A1}}{r} v_\theta, \quad \Phi_{H2} = -\frac{u_{\theta A1}^2}{2r^2} + \frac{u_{\theta A1}^2}{10}\left(1 - \frac{1}{r^2}\right) v^2 + \frac{u_{\theta A1}^2}{2r^2} v_\theta^2$$

and $c_3(r) = \frac{u_3}{r}$, for some constant $u_3 \neq 0$. Moreover, (2.5) and (2.12) give that $\Phi_{W1}(Y, v) = z_1(Y) v_\theta$, for some function z_1, and $\Phi_{Wi}, i \geqslant 2$ split into a hydrodynamical part $x_i(Y) + y_i(Y) v^2 + z_i(Y) v_\theta + t_i(Y) v_r$ and a non-hydrodynamic part involving Hilbert terms of lower order than i. Notice that Φ_{W4} is the sum of $z_1' v_\theta v_r \bar{B}$ and a polynomial in the v-variable with bounded coefficients in the r-variable. More precisely,

$$\begin{aligned}
\Phi_{W2} &= x_2 + y_2 v^2 + z_2 v_\theta + \left(b_1(r_B) z_1 + \frac{1}{2} z_1^2 \right) v_\theta^2, \\
\Phi_{W3} &= x_3 + y_3 v^2 + z_3 v_\theta + t_3 v_r + (b_1(r_B) y_2 + z_1 y_2 + z_1 d_2(r_B)) v_\theta v^2 \\
&\quad + (b_1(r_B) z_2 + z_1 z_2 + z_1 b_2(r_B) + Y b_1'(r_B) z_1) v_\theta^2 \\
&\quad + \left(\frac{1}{2} b_1^2(r_B) z_1 + \frac{1}{2} b_1(r_B) z_1^2 + \frac{1}{6} z_1^3 \right) v_\theta^3, \\
\Phi_{W4} &= x_4 + y_4 v^2 + z_4 v_\theta + t_4 v_r + z_1' v_r v_\theta \bar{B}(v) + \cdots \quad .
\end{aligned}$$

Equations (2.13) have solutions if and only if the following compatibility conditions hold,

$$\begin{aligned}
\int \Bigg(v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} &+ \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i \left(\frac{Y}{r_B} \right)^i N(\Phi_{Hk-4-i}(r_B, .) \\
&+ \Phi_{Wk-4-i} \Bigg) (v^2 - 5, v_\theta) M(v) dv = 0, \quad k \geqslant 5, \qquad (2.19)
\end{aligned}$$

and

$$\begin{aligned}
\int \Bigg(v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} &+ \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i \left(\frac{Y}{r_B} \right)^i N(\Phi_{Hk-4-i}(r_B, .) \\
&+ \Phi_{Wk-4-i} \Bigg) (1, v_r) M(v) dv = 0, \quad k \geqslant 5. \qquad (2.20)
\end{aligned}$$

Equations (2.19) (resp. (2.20)) provide second-order (resp. first-order) differential equations for y_i and z_i (resp. $x_i + 5y_i$ and t_i). In particular,

$$\begin{gathered}
w_1 z_1'' - \tfrac{u_3}{r_B} z_1' = 0, \\
\left(x_2 + 5y_2 + b_1(r_B) z_1 + \tfrac{1}{2} z_1^2 \right)' = 0, \\
w_2 y_2'' + \tfrac{10}{r_B} y_2' + A_1 = 0, \quad w_1 z_2'' - \tfrac{u_3}{r_B} z_2' + A_1 = 0, \qquad (2.21) \\
t_3' = 0, \\
(x_3 + 5y_3 + b_1(r_B) z_2 + z_1 z_2 + z_1 b_2(r_B) + Y b_1'(r_B) z_1)' = \tfrac{1}{r_B} (2 b_1(r_B) z_1 + z_1^2), \\
w_2 y_3'' + \tfrac{10}{r_B} y_3' + A_2 = 0, \\
w_1 z_3'' - \tfrac{u_3}{r_B} z_3' + ((b_1(r_B) + z_1)(c_5(r_B) + t_5))' + A_2 = 0, \\
t_4' + \tfrac{1}{r_B} (t_3 + c_3(r_B)) + c_3'(r_B) = 0, \qquad (2.22) \\
(x_4 + 5y_4)' + A_3 = 0.
\end{gathered}$$

Here $A_i, 1 \leqslant i \leqslant 3$, denote terms involving Hilbert and suction coefficients up to order i. Together with the boundary conditions (2.3) at first and second orders, and the conditions (2.6) and (1.4), this fixes

$$\Phi_{W1}(Y, v) = \left(u_{\theta B1} - \frac{u_{\theta A1}}{r_B}\right) e^{\frac{u_3 Y}{w_1 r_B}} \cdot v_\theta,$$

as well as Φ_{W2} in terms of u_3, and implies that $t_3 = t_4 = 0$. Then, giving the value 0 to any coefficient of order bigger than 5 in the second-order differential equations satisfied by y_i and z_i, $3 \leqslant i \leq 4$ and taking into account (2.3–6) fixes the functions y_i and z_i, $3 \leqslant i \leqslant 4$ in terms of u_i. Finally the Knudsen analysis at third and fourth orders in Lemma 2.1–2 below, makes the first-order differential equations satisfied by $x_3 + 5y_3$ and $x_4 + 5y_4$ compatible with (2.3) and (2.6) at third and fourth orders. ∎

Lemma 2.1. Set $\eta = \frac{r-1}{\epsilon^4}$, $\mu = \frac{r - r_B}{\epsilon^4}$. There are unique Knudsen boundary layers $\Phi_{K3A}(\eta, v)$ and $\Phi_{K3B}(\mu, v)$, and boundary values $\Phi_{H3}(1, v)$ and $\Phi_{W3}(0, v)$, such that

$$\begin{aligned} v_r \frac{\partial \Phi_{K3A}}{\partial \eta} &= \tilde{L}\Phi_{K3A}, \quad \eta > 0, \quad v \in \mathbb{R}^3, \\ \Phi_{K3A}(0, v) &= \Phi_{A3}(v) - \Phi_{H3}(1, v), \quad v_r > 0, \\ \lim_{\eta \to +\infty} \Phi_{K3A}(\eta, v) &= 0, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} v_r \frac{\partial \Phi_{K3B}}{\partial \mu} &= \tilde{L}\Phi_{K3B}, \quad \mu < 0, \quad v \in \mathbb{R}^3, \\ \Phi_{K3B}(0, v) &= \Phi_{B3}(v) - \Phi_{H3}(r_B, v) - \Phi_{W3}(0, v), \quad v_r < 0, \\ \lim_{\mu \to -\infty} \Phi_{K3B}(\mu, v) &= 0, \end{aligned} \tag{2.24}$$

with the limits in $\tilde{L}^q$-sense. The boundary layers fix the possible values of $a_3(1)$, $d_3(1)$, u_3, $b_3(1)$ and $x_3(0)$, $y_3(0)$, $z_3(0)$, hence complete the definitions of Φ_{H3} and Φ_{W3}.

Proof of Lemma 2.1. The function

$$\psi_{K3A}(\eta, v) := \Phi_{K3A}(\eta, v) - u_3(g - A - Dv^2 - Ev_\theta - v_r),$$

with g, A, D and E defined in (2.8–9) and u_3 still unknown, should satisfy

$$\begin{aligned}
v_r \frac{\partial \psi_{K3A}}{\partial \eta} &= \tilde{L}\psi_{K3A}, \quad \eta > 0, \ \ v \in \mathbb{R}^3, \\
\psi_{K3A}(0, v) &= u_3 A - a_3(1) + (u_3 D - d_3(1))v^2 \\
&\quad + \left(u_3 E - \frac{1}{2} u_{\theta A1}^3 - b_3(1) \right) v_\theta, \quad v_r > 0, \\
\lim_{\eta \to +\infty} \psi_{K3A}(\eta, v) &= 0.
\end{aligned}$$

Hence,

$$a_3(1) = u_3 A, \quad d_3(1) = u_3 D, \quad b_3(1) = u_3 E - \frac{1}{2} u_{\theta A1}^3, \quad \psi_{K3A} = 0,$$

so that

$$\Phi_{K3A} = u_3 (g - A - Dv^2 - E v_\theta - v_r).$$

Analogously, the function

$$\psi_{K3B}(\mu, v) := \Phi_{K3B}(\mu, v) - \frac{u_3}{r_B}(g(-\mu, -v) - A - Dv^2 + E v_\theta + v_r),$$

should satisfy

$$\begin{aligned}
v_r \frac{\partial \psi_{K3B}}{\partial \mu} &= \tilde{L}\psi_{K3B}, \quad \mu < 0, \quad v \in \mathbb{R}^3, \\
\psi_{K3B}(0, v) &= \Delta - \frac{u_3}{r_B} A - a_3(r_B) - x_3(0) - \left(\frac{u_3}{r_B} D + d_3(r_B) + y_3(0) \right) v^2 \\
&\quad + \left(u_{\theta B1} \left(\frac{r_B^2 - 1}{r_B^2} u_{\theta A1}^2 - \frac{7}{2} \tau_{B2} - u_{\theta B1}^2 \right) \right. \\
&\quad \left. + \frac{u_3}{r_B} E - b_3(r_B) - z_3(0) \right) v_\theta, \quad v_r < 0, \\
\lim_{\mu \to -\infty} \psi_{K3B}(\mu, v) &= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&x_3(0) = \Delta - \frac{u_3}{r_B} A - a_3(r_B), \quad y_3(0) = -\frac{u_3}{r_B} D - d_3(r_B), \\
&z_3(0) = -u_{\theta B1} \left(\frac{r_B^2 - 1}{r_B^2} u_{\theta A1}^2 - \frac{7}{2} \tau_{B2} - u_{\theta B1}^2 \right) + \frac{u_3}{r_B} E - b_3(r_B), \quad \psi_{K3B} = 0,
\end{aligned}$$

and

$$\Phi_{K3B}(\mu, v) = \frac{u_3}{r_B}(g(-\mu, -v) - A - Dv^2 + Ev_\theta + v_r).$$

Moreover, by integration of (2.21) and (2.16) on $(-\infty, 0)$ and $(1, r_B)$ respectively,

$$x_3(0) + 5y_3(0) = \frac{w_1}{2r_B^2 u_3}(u_{\theta B1}r_B + 3u_{\theta A1}r_B)(u_{\theta B1}r_B - u_{\theta A1}),$$
$$x_3(0) + 5y_3(0) = \Delta - u_3(A + 5D)\left(\frac{1}{r_B} + 1\right).$$

And so, u_3 must solve the equation

$$u_3^2(A + 5D)\frac{r_B + 1}{r_B} - \Delta u_3 + \frac{w_1}{2r_B^2}(3u_{\theta A1} + u_{\theta B1}r_B)(u_{\theta A1} - u_{\theta B1}r_B) = 0. \tag{2.25}$$

The pointwise estimates in refs. 8 and 14 imply the $\tilde{L}^q$-version of (2.23–24). ▮

End of Proof of Proposition 2.1. A study of the positive roots u_3 to (2.25) leads to the three cases described in Proposition 2.1 for Δ with respect to Δ_{bif}. That proof requires a nondegeneracy in the Milne asymptotics (2.9),

$$A + 5D < 0. \tag{2.26}$$

The condition is expected to hold on physical grounds and has been verified numerically for hard spheres and Maxwellian molecules. In this paper it is required to hold for the kernels B, precisely when the bifurcation situation is being considered. A mathematical proof of (2.26) related to the numerical approach, seems feasible but has not been undertaken here. Our aim is merely to illustrate that the present setup also covers bifurcation situations. Instead a separate paper under preparation will be devoted to a fundamental bifurcation problem using our approach, namely the Taylor–Couette bifurcation for the two-roll setup of ref. 27 with axial dependence. We want to stress that the condition (2.26) is not used to obtain the existence of isolated or multiple solutions, but only to enter the bifurcation situation discussed in Theorem 1.2. ▮

Lemma 2.2. Set $\eta=\frac{r-1}{\epsilon^4}$, $\mu=\frac{r-r_B}{\epsilon^4}$. There are unique Knudsen boundary layers $\Phi_{K4A}(\eta,v)$ and $\Phi_{K4B}(\mu,v)$, and boundary values $\Phi_{H4}(1,v)$ and $\Phi_{W4}(0,v)$ such that

$$\begin{aligned}
v_r\frac{\partial\Phi_{K4A}}{\partial\eta} &= \tilde{L}\Phi_{K4A}+2\tilde{J}(\Phi_{H1}(1),\Phi_{K3A})), \quad \eta>0,\ v\in\mathbb{R}^3,\\
\Phi_{K4A}(0,v) &= \Phi_{A4}(v)-\Phi_{H4}(1,v), \quad v_r>0,\\
\lim_{\eta\to+\infty}\Phi_{K4A} &= 0,
\end{aligned}$$

and

$$\begin{aligned}
v_r\frac{\partial\Phi_{K4B}}{\partial\mu} &= \tilde{L}\Phi_{K4B}+2\tilde{J}(\Phi_{H1}(r_B)+\Phi_{W1}(0),\Phi_{K3B}), \quad \mu<0,\ v\in\mathbb{R}^3,\\
\Phi_{K4B}(0,v) &= \Phi_{B4}(v)-\Phi_{H4}(r_B,v)-\Phi_{W4}(0,v), \ \ v_r<0,\\
\lim_{\mu\to-\infty}\Phi_{K4B} &= 0,
\end{aligned}$$

with the limits in $\tilde{L}^q$-sense. The fourth order Knudsen boundary layers fix the possible values of $a_4(1)$, $d_4(1)$, $u_4=r_Bc_4(r_B)$ and $x_4(0)$, $y_4(0)$, $z_4(0)$, hence complete the definitions of Φ_{H4} and Φ_{W4}.

Proof of Lemma 2.2. Analogously to ref. 8, there are unique solutions α and β to

$$\begin{aligned}
v_r\frac{\partial\alpha}{\partial\eta} &= \tilde{L}\alpha+2\tilde{J}(\Phi_{H1}(1),\Phi_{K3A}), \quad \eta>0,\ \ v\in I\!R^3,\\
\alpha(0,v) &= -u_{\theta A1}\left(u_3Dv_\theta v^2+\left(u_3E+\frac{1}{4}u_{\theta A1}^3\right)v_\theta^2+u_3v_rv_\theta\right), \quad v_r>0,\\
&\int v_r\alpha(\eta,v)M(v)dv=0,
\end{aligned}$$

and

$$\begin{aligned}
v_r\frac{\partial\beta}{\partial\eta} &= \tilde{L}\beta+2\tilde{J}(\Phi_{H1}(r_B,-v)+\Phi_{W1}(0,-v),\Phi_{K3B}(-\eta,-v)), \quad \eta>0,\ v\in\mathbb{R}^3,\\
\beta(0,v) &= \Phi_{B4}(-v)-\left(\Phi_{H4}(r_B,-v)-a_4(r_B)-d_4(r_B)v^2-b_4(r_B)v_\theta-\frac{u_4}{r_B}v_r\right)\\
&\quad -(\Phi_{W4}(0,-v)-x_4(0)-y_4(0)v^2-z_4(0)v_\theta), \ \ v_r>0,\\
&\int v_r\beta(\eta,v)M(v)dv=0.
\end{aligned}$$

Moreover,

$$\alpha \in Ker\tilde{L}, \quad \beta \in Ker\tilde{L}^{\perp},$$
$$\lim_{\eta\to+\infty} \alpha(\eta, v)=a_\infty+d_\infty v^2+b_\infty v_\theta, \quad \lim_{\eta\to+\infty} \beta(\eta, v)=r_\infty+s_\infty v^2+t_\infty v_\theta,$$

for some constants a_∞, d_∞, b_∞, r_∞, s_∞ and t_∞. The function

$$\psi_{K4A}(\eta, v) := \Phi_{K4A}(\eta, v) - u_4(g - A - Dv^2 - Ev_\theta - v_r)$$
$$-(\alpha - a_\infty - d_\infty v^2 - b_\infty v_\theta)$$

should satisfy

$$v_r \frac{\partial \psi_{K4A}}{\partial \eta} = \tilde{L}\psi_{K4A}, \quad \eta > 0, \ \ v \in \mathbb{R}^3,$$
$$\psi_{K4A}(0, v) = \frac{1}{8}u_{\theta A1}^2 + a_\infty + u_4A - a_4(1) + (d_\infty + u_4D - d_4(1))v^2$$
$$+(b_\infty + u_4E - b_4(1))v_\theta, \quad v_r < 0,$$
$$\lim_{\mu\to-\infty} \psi_{K4A} = 0.$$

Hence,

$$a_4(1) = \frac{1}{8}u_{\theta A1}^2 + a_\infty + u_4A, \quad d_4(1) = d_\infty + u_4D, \quad b_4(1) = b_\infty + u_4E,$$
$$\psi_{K4A} = 0,$$

so that

$$\Phi_{K4A} = \alpha - a_\infty - d_\infty v^2 - b_\infty v_\theta + u_4(g - A - Dv^2 - Ev_\theta - v_r).$$

Analogously, the function

$$\psi_{K4B}(\mu, v) := \Phi_{K4B}(\mu, v) - \frac{u_4}{r_B}(g(-\mu, -v) - A - Dv^2 + Ev_\theta - v_r)$$
$$-(\beta(-\mu, -v) - r_\infty - s_\infty v^2 + t_\infty v_\theta)$$

should satisfy

$$
\begin{aligned}
v_r \frac{\partial \psi_{K4B}}{\partial \mu} &= \tilde{L}\psi_{K4B}, \quad \mu < 0, \quad v \in \mathbb{R}^3, \\
\psi_{K4B}(0, v) &= r_\infty + \frac{u_4}{r_B} A - a_4(r_B) - x_4(0) \\
&\quad + \left(s_\infty + \frac{u_4}{r_B} D - d_4(r_B) - y_4(0) \right) v^2 \\
&\quad - \left(t_\infty + \frac{u_4}{r_B} E + b_4(r_B) + z_4(0) \right) v_\theta, \quad v_r < 0, \\
\lim_{\mu \to -\infty} \psi_{K4B}(\mu, v) &= 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&x_4(0) = r_\infty - a_4(r_B) + u_4 \frac{A}{r_B}, \quad y_4(0) = s_\infty - d_4(r_B) + u_4 \frac{D}{r_B}, \\
&z_4(0) = t_\infty - b_4(r_B) + u_4 \frac{E}{r_B}, \quad \psi_{K4B} = 0,
\end{aligned}
$$

so that

$$
\begin{aligned}
\Phi_{K4B}(\mu, v) = \beta(-\mu, -v) - r_\infty - s_\infty v^2 + t_\infty v_\theta \\
+ \frac{u_4}{r_B}(g(-\mu, -v) - A - Dv^2 + Ev_\theta).
\end{aligned}
$$

Moreover, by integration of (2.22) and (2.18) on $(-\infty, 0)$ and $(1, r_B)$ respectively,

$$
(x_4 + 5y_4)(0) = \bar{A}_3, \quad (a_4 + 5d_4)(r_B) = u_4(A + 5D) + \tilde{A}_3,
$$

where $\bar{A}_3$ and $\tilde{A}_3$ are given in terms of up to third order coefficients. This fixes the value of u_4, hence uniquely defines Φ_{K4A} and Φ_{K4B}. ∎

Lemma 2.3. Denote by $l := \frac{1}{\epsilon^4}\left(\tilde{L}\varphi + \tilde{J}(\varphi, \varphi) - \epsilon^4 D\varphi\right)$. Then,

$$
|\, l\,|_q := \left(\int M(v) \left(\int |\, l(x, v)\,|^q \, dx \right)^{\frac{2}{q}} dv \right)^{\frac{1}{2}}
$$

is of order one in $\tilde{L}^q$ with respect to ϵ.

Proof of Lemma 2.3. By definition of φ,

$$
\begin{aligned}
\frac{\epsilon^2}{2} l = &\tilde{J}(\Phi_{H1} - \Phi_{H1}(r_B), \Phi_{W1})\\
&+\epsilon\Big(\tilde{J}(\Phi_{H1} - \Phi_{H1}(r_B), \Phi_{W2}) + \tilde{J}(\Phi_{W1}, \Phi_{H2} - \Phi_{H2}(r_B) - Y\Phi'_{H1}(r_B)\Big)\\
&+\epsilon^2\Big(\tilde{J}(\Phi_{W3}, \Phi_{H1} - \Phi_{H1}(r_B)) + \tilde{J}(\Phi_{W2}, \Phi_{H2} - \Phi_{H2}(r_B) - Y\Phi'_{H1}(r_B))\\
&\quad + \tilde{J}(\Phi_{W1}, \Phi_{H3} - \Phi_{H3}(r_B) - Y\Phi'_{H2}(r_B) - \frac{Y^2}{2}\Phi''_{H1}(r_B))\\
&\quad + \tilde{J}(\Phi_{K3A}, \Phi_{H1} - \Phi_{H1}(1) + \Phi_{W1}) + \tilde{J}(\Phi_{K3B}, \Phi_{H1} - \Phi_{H1}(r_B)\\
&\quad + \Phi_{W1} - \Phi_{W1}(0))\Big) + O(\epsilon^3).
\end{aligned}
$$

Hence

$$
\begin{aligned}
l = \epsilon\tilde{J}\Big(&\gamma_1(r)Y^3\Phi_{W1} + \gamma_2(r)Y^2\Phi_{W2} + \gamma_3(r)Y\Phi_{W3} + \gamma_4(r)Y\Phi_{K3A}\\
&+\gamma_5(r)Y\Phi_{K3B}, v_\theta\Big) + \tilde{J}(\Phi_{K3A}, \Phi_{W1}) + O(\epsilon),
\end{aligned}
$$

where $(\gamma_i)_{1\leqslant i\leqslant 5}$ are bounded functions in r and $\eta = \frac{r-1}{\epsilon^4}$. We notice that Φ_{W1} is exponentially small near r_A. From here the ϵ-bound for l follows from the decay properties of Φ_{Wj}, $j = 1, \ldots, 4$ in the suction layer, and of $\Phi_{Ki,j}$, $i = 3, 4$, $j = A, B$, in the Knudsen layer. ∎

3. ON THE CONTROL OF $f_\perp$ AND $f_\parallel$

We take $\psi_0 = 1$, $\psi_\theta = v_\theta$, $\psi_r = v_r$, $\psi_z = v_z$, $\psi_4 = \frac{1}{\sqrt{6}}(v^2 - 3)$ as an orthonormal basis for the kernel of $\tilde{L}$ in $L^2_M(\mathbb{R}^3)$. Recall that in this paper all functions are even in v_z. For functions $f \in L^2_M([r_A, r_B] \times \mathbb{R}^3)$ we shall use the earlier splitting into $f = f_\parallel + f_\perp = P_0 f + (I - P_0)f$, such that

$$
\begin{aligned}
&f_\parallel(r, v) = f_0(r) - \frac{\sqrt{6}}{2} f_4(r) + f_\theta(r)v_\theta + f_r(r)v_r + \frac{\sqrt{6}}{6} f_4(r)v^2,\\
&\int M(v)(1, v, v^2) f_\perp(r, v)dv = 0,\\
&\int M\psi_0 f(r, v)dv = f_0(r), \quad \int M\psi_4 f(r, v)dv = f_4(r),\\
&\int M\psi_\theta f(r, v)dv = f_\theta(r), \quad \int M\psi_r f(r, v)dv = f_r(r).
\end{aligned}
$$

The ψ_z-moment of $f_\|$ vanishes, since f is even in v_z. Set $\tilde{\nu} := \nu\epsilon^4$, and $Df := v_r \frac{\partial f}{\partial r} + \frac{1}{r} Nf$ with N given by (1.3). Due to the symmetries in the present setup, the position space may be changed from $I\!R^2$ with measure dx, to $I\!R^+$ with measure rdr. The relevant boundary space becomes

$$L^+ := \left\{ f;\, |f|_\sim = \left(\int_{v_r>0} v_r M(v) \,|\, f(r_A, v) \,|^2 \, dv \right)^{\frac{1}{2}} \right.$$
$$\left. + \left(\int_{v_r<0} |\, v_r \,|\, M(v) \,|\, f(r_B, v) \,|^2 \, dv \right)^{\frac{1}{2}} < +\infty \right\}.$$

We shall also use

$$\mathcal{W}^{q-}([r_A, r_B] \times \mathbb{R}^3) = \mathcal{W}^{q-} := \{ f;\, \nu^{\frac{1}{2}} f \in \tilde{L}^q,\, \nu^{-\frac{1}{2}} Df \in \tilde{L}^q,\, \gamma^+ f \in L^+ \}.$$

Define

$$f_{\theta^i r^j}(r) := \int M v_\theta^i v_r^j f_\perp(r, v) dv, \quad i + j \geqslant 2,$$

and $f_{\theta^i r^j 2}(r)$ correspondingly, when there is an extra factor $|\, v \,|^2$ in the integrand.

The main *a priori* estimates will below be given in $\tilde{L}^2$. We shall require that $|u_{\theta A1}|$, $|u_{\theta B1}|$ and $|\tau_{B2}|$ are bounded by some value δ', which implies that the coefficients in the individual terms for φ^j as given in Section 2, $j = 1, \ldots, 4$, are bounded by some multiple of δ'. When the Knudsen number is ϵ^m and $m > 2$, in order that the $\tilde{L}^2$-approach becomes sharp enough for the intended applications, a preliminary reorganization is first performed on the original linearized problem

$$DF = \frac{1}{\epsilon^m} \left(\tilde{L} F + \sum_{j=1}^{j_1} \epsilon^j \tilde{J}(F, \varphi^j) + g \right), \quad F_{/\partial\Omega^+} = F_b. \tag{3.1}$$

This is related to the velocity perturbations of order m-th root of the Knudsen number, becoming stronger in relation to the Knudsen number with increasing m. We carry out the procedure for the case $m = 4$. Some terms will be moved from $\epsilon^{-3} \tilde{J}(F, \varphi^1)$ in (3.1) to the $\epsilon^{-4} \tilde{L} F$-term, together with follow-up changes in other terms in order to move certain couplings between moments from lower to higher order terms. This will be important in the control of the outgoing fluxes in Proposition 3.2. With w_1 and u_3 as defined in the previous section, set

$$
\begin{aligned}
k_4 &:= \int v_r^2 \psi_4 \bar{A} M dv, \quad k_5 := \int v_r \tilde{J}(\psi_4, v_r) \bar{A} M dv, \\
k_6 &:= \int v_r v_\theta \tilde{J}(v_\theta, v_r) \bar{B} M dv, \quad c := \frac{k_5 u_3}{k_4}, \quad d := \frac{k_6 u_3}{w_1}, \\
m_4 &:= k_4^{-1} \left(\frac{(\frac{r}{r_A})^{\frac{c}{\epsilon}} - 1}{(\frac{r_B}{r_A})^{\frac{c}{\epsilon}} - 1} \tilde{F}_4(r_B) + \frac{(\frac{r}{r_B})^{\frac{c}{\epsilon}} - 1}{(\frac{r_A}{r_B})^{\frac{c}{\epsilon}} - 1} \tilde{F}_4(r_A) \right), \\
m_\theta &: = w_1^{-1} \left(\frac{(\frac{r}{r_A})^{\frac{d}{\epsilon}} - 1}{(\frac{r_B}{r_A})^{\frac{d}{\epsilon}} - 1} \tilde{F}_\theta(r_B) + \frac{(\frac{r}{r_B})^{\frac{d}{\epsilon}} - 1}{(\frac{r_A}{r_B})^{\frac{d}{\epsilon}} - 1} \tilde{F}_\theta(r_A) \right).
\end{aligned}
$$

Lemma 3.1. A solution F of (3.1) can be split into the sum of a function F_- and ϵ times a nonhydrodynamic linear combination of $F_r(1)$, m_4 and m_θ, with $F_\parallel = F_{-\parallel}$ and F_- solution to the equation

$$
\begin{aligned}
DF_- = {} & \frac{1}{\epsilon^4} \tilde{L} F_- + \frac{1}{\epsilon^3} \tilde{J} \left(F_- - \frac{F_r(1)}{r} v_r - m_\theta v_\theta - m_4 \psi_4, \varphi^1 \right) \\
& + \sum_{j=2}^{4} \epsilon^{j-4} \tilde{J} \left(F_- - \frac{F_r(1)}{r} v_r, \varphi^j \right) + \sum_{j=5}^{j_1} \epsilon^{j-4} \tilde{J}(F_-, \varphi^j) \\
& + \frac{1}{\epsilon^4} g + \epsilon (F_r(1) \beta_1 + m_\theta \beta_2 + m_4 \beta_3),
\end{aligned} \tag{3.2}
$$

where β_i, $1 \leqslant i \leqslant 3$, are known functions in nonnegative powers of ϵ.

Proof of Lemma 3.1. Equation (3.1) can also be written as

$$
\begin{aligned}
DF = {} & \frac{1}{\epsilon^4} \tilde{L} \left(F - \epsilon \frac{c(r)}{2} \left(\frac{F_r(1)}{r} v_r v_\theta + m_\theta (v_\theta^2 - 1) + \frac{m_4}{\sqrt{6}} v_\theta (v^2 - 5) \right) \right) \\
& + \frac{1}{\epsilon^3} \tilde{J} \left(F - \frac{F_r(1)}{r} v_r - m_\theta v_\theta - m_4 \psi_4, \varphi^1 \right) \\
& + \sum_{j=2}^{j_1} \epsilon^{j-4} \tilde{J}(F, \varphi^j) + \frac{1}{\epsilon^4} g \\
= {} & \frac{1}{\epsilon^4} \tilde{L} \left(F - \epsilon \frac{c(r)}{2} \left(\frac{F_r(1)}{r} v_r v_\theta + m_\theta (v_\theta^2 - 1) + \frac{m_4}{\sqrt{6}} v_\theta (v^2 - 5) \right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^2\left(\frac{F_r(1)}{r}\alpha_1+m_\theta\alpha_2+m_4\alpha_3\right)\Bigg)+\frac{1}{\epsilon^3}\tilde{J}\left(F-\frac{F_r(1)}{r}v_r-m_\theta v_\theta-m_4\psi_4\right.\\
&\left.-\epsilon\frac{c(r)}{2}\left(\frac{F_r(1)}{2}v_r v_\theta+m_\theta(v_\theta^2-1)+\frac{m_4}{\sqrt{6}}v_\theta(v^2-5)\right),\varphi^1\right)\\
&+\sum_{j=2}^{j_1}\epsilon^{j-4}\tilde{J}(F,\varphi^j)+\frac{1}{\epsilon^4}g,
\end{aligned}
$$

where

$$
\tilde{L}(\alpha_1)=c_1\tilde{J}(v_r v_\theta, v_\theta),\quad \tilde{L}(\alpha_2)=c_2\tilde{J}(v_\theta^2, v_\theta),\quad \tilde{L}(\alpha_3)=c_3\tilde{J}\left(\frac{v_\theta(v^2-5)}{\sqrt{6}}, v_\theta\right).
$$

Continuing the same way one gets the equation

$$
\begin{aligned}
DF=&\frac{1}{\epsilon^4}\tilde{L}F_-+\frac{1}{\epsilon^3}\tilde{J}\left(F_--\frac{F_r(1)}{r}v_r-m_\theta v_\theta-m_4\psi_4,\varphi^1\right)\\
&+\sum_{j=2}^{4}\epsilon^{j-4}\tilde{J}\left(F_--\frac{F_r(1)}{r}v_r,\varphi^j\right)+\sum_{j=5}^{j_1}\epsilon^{j-4}\tilde{J}(F,\varphi^j)+\frac{1}{\epsilon^4}g+\epsilon J_1,
\end{aligned}
$$

where J_1 is a nonhydrodynamic linear combination of $F_r(1)$, m_θ and m_4, $F_{-\parallel}=F_\parallel$ and $F_{-\perp}$ is the sum of $F_\perp$ and a nonhydrodynamic linear combination of $F_r(1)$, m_θ and m_4. And so, writing DF as the sum of DF_- and known terms leads to the Equation (3.2). ∎

Lemma 3.2. Let $\tilde{F}_{-4}:=k_4F_4+F_{-r^2\bar{A}}$ and $\tilde{F}_{-\theta}:=w_1F_\theta+F_{-\theta r^2\bar{B}}$. Then

$$
\begin{aligned}
\tilde{F}_{-4}(r)=&\,m_4(r)+\int_1^r\frac{(\frac{r_B}{s})^{\frac{k_5u_3}{k_4}}-(\frac{r}{s})^{\frac{k_5u_3}{k_4}}}{r_B^{\frac{k_5u_3}{k_4}}-1}G_4(s)ds\\
&+\int_r^{r_B}\frac{(\frac{r}{s})^{\frac{k_5u_3}{k_4}}-(\frac{1}{s})^{\frac{k_5u_3}{k_4}}}{(\frac{1}{r_B})^{\frac{k_5u_3}{k_4}}-1}G_4(s)ds,\\
\tilde{F}_{-\theta}(r)=&\,m_\theta(r)+\int_1^r\frac{(\frac{r_B}{s})^{\frac{k_6u_3}{w_1}}-(\frac{r}{s})^{\frac{k_6u_3}{w_1}}}{r_B^{\frac{k_6u_3}{w_1}}-1}G_\theta(s)ds\\
&+\int_r^{r_B}\frac{(\frac{r}{s})^{\frac{k_6u_3}{w_1}}-(\frac{1}{s})^{\frac{k_6u_3}{w_1}}}{(\frac{1}{r_B})^{\frac{k_6u_3}{w_1}}-1}G_\theta(s)ds,
\end{aligned}
$$

where G_4 and G_θ satisfy

$$\begin{aligned}|G_4|_2+|G_\theta|_2\leqslant c\Big(&\|F_{-\perp}\|_2+\epsilon\delta'\|F_\|\|_2+\frac{1}{\epsilon^8}|g_\||_2+\frac{1}{\epsilon^4}|g_\perp|_2\\&+\epsilon(|F_r(1)|+|m_\theta|_2+|m_4|_2)\Big).\end{aligned}\tag{3.3}$$

Proof of Lemma 3.2. A multiplication of (3.2) with $v_\theta M$ (resp. v^2M) and integration over $I\!R_v^3$ leads to

$$\begin{aligned}F_{-\theta r}(r)&=\frac{F_{-\theta r}(1)}{r^2}+\frac{1}{r^2}\int_1^r s^2\frac{g_\theta}{\epsilon^4}ds+\mathcal{O}(\epsilon),\\F_{-r2}(r)&=\frac{F_{-r2}(1)}{r}+\frac{1}{r\epsilon^4}\int_1^r s\left(\sqrt{6}g_4+3g_0\right)ds+\mathcal{O}(\epsilon).\end{aligned}$$

Multiply equation (3.2) with $\bar{A}(|v|)v_rM$ and integrate over $\mathbb{R}_v^3$,

$$\begin{aligned}\left(\int v_r^2\bar{A}MF_-dv\right)'&=\left(k_4F_4+F_{-r^2\bar{A}}\right)'=\frac{1}{r}\left(F_{-\theta^2\bar{A}}-F_{-r^2\bar{A}}\right)\\&+\frac{1}{\epsilon^4}\Big(\frac{c_{r2}}{r}+\frac{1}{r\epsilon^4}\int_1^r s(\sqrt{6}g_4-2g_0)ds+\int v_r\bar{A}\tilde{J}\Big(F_{-\perp},\sum_1^4\epsilon^j\varphi^j\Big)Mdv\\&\quad+\epsilon^3F_4(c_3+\epsilon c_4)k_5+\epsilon^4F_\theta b_1c_3\int v_r\bar{A}\tilde{J}(v_\theta,v_rv_\theta)Mdv\Big)\\&+\sum_{j=5}^{j_1}\epsilon^{j-4}\int v_r\bar{A}\tilde{J}(F_-,\varphi^j)Mdv+\frac{1}{\epsilon^4}\int gv_r\bar{A}Mdv\\&+\epsilon\int v_r\bar{A}(F_r(1)\gamma_1+m_\theta\gamma_2+m_4\gamma_3)Mdv.\end{aligned}\tag{3.4}$$

Here γ_i, $i=1,2,3$, are known functions in positive powers of ϵ. Using the spectral inequality, we notice that

$$\begin{aligned}k_4&=\int v_r^2\psi_4\bar{A}Mdv=\frac{1}{\sqrt{6}}\int v_rv^2v_r\bar{A}Mdv\\&=\frac{1}{\sqrt{6}}\int v_r(v^2-5)v_r\bar{A}Mdv=\frac{1}{\sqrt{6}}\int\tilde{L}(v_r\bar{A})v_r\bar{A}Mdv\\&<-c\int|v_r\bar{A}|^2Mdv<0.\end{aligned}$$

In (3.4) c_3 and c_4 respectively denote the coefficients of v_r in φ^3 and φ^4. Then $c_3=\frac{u_3}{r}$, with $u_3>0$ in the present case. Its coefficient in the ϵ^{-1}-term of (3.4) is $F_4k_5 = F_4 \int \tilde{J}(\psi_4, v_r)v_r\bar{A}Mdv$, where

$$
\begin{aligned}
-\int \tilde{J}(\psi_4, v_r)v_r\bar{A}Mdv &= \frac{1}{\sqrt{6}}\int \tilde{L}v_r v^2 \cdot v_r\bar{A}Mdv = \frac{1}{\sqrt{6}}\int \tilde{L}v_r(v^2-5)\cdot v_r\bar{A}Mdv \\
&= \frac{1}{\sqrt{6}}\int v_r(v^2-5)\cdot \tilde{L}v_r\bar{A}\cdot Mdv \\
&= \frac{1}{\sqrt{6}}\int |v_r(v^2-5)|^2 Mdv > 0.
\end{aligned}
$$

Hence $k_4k_5 > 0$.

Let $\tilde{F}_{-4} = k_4F_4 + F_{-r^2\bar{A}}$. In (3.4) regroup the terms as

$$
\begin{aligned}
\tilde{F}'_{-4} - \frac{k_5u_3}{k_4r\epsilon}\tilde{F}_{-4} &= \frac{c_{r2}}{r\epsilon^4} + \Bigg\{\frac{1}{r}\Big(F_{-\theta^2\bar{A}} - F_{-r^2\bar{A}}\Big) \\
&\quad + \frac{1}{\epsilon^4}\Bigg(\frac{1}{r\epsilon^4}\int_1^r s(\sqrt{6}g_4 - 2g_0)ds \\
&\qquad + \int v_r\bar{A}\tilde{J}\Big(F_{-\perp}, \sum_1^4 \epsilon^j\varphi^j\Big)Mdv\Bigg) \\
&\quad - \frac{k_5u_3}{k_4r\epsilon}F_{-r^2\bar{A}} + F_4c_4k_5 + F_\theta b_1c_3\int v_r\bar{A}\tilde{J}(v_\theta, v_rv_\theta)Mdv \\
&\quad + \sum_{j=5}^{j_1}\epsilon^{j-4}\int v_r\bar{A}\tilde{J}(F_-, \varphi^j)Mdv + \frac{1}{\epsilon^4}\int gv_r\bar{A}Mdv \\
&\quad + \epsilon\int v_r\bar{A}(F_r(1)\gamma_1 + m_\theta\gamma_2 + m_4\gamma_3)Mdv\Bigg\}.
\end{aligned}
$$

Here denoting the expression within $\Big\{...\Big\}$ by G_4 and setting $c := \frac{k_5u_3}{k_4}$, gives $c > 0$ and

$$
\Big(\tilde{F}_{-4}r^{-\frac{c}{\epsilon}}\Big)' = \frac{c_{r2}}{\epsilon^4}r^{-\frac{c}{\epsilon}-1} + G_4r^{-\frac{c}{\epsilon}}.
$$

This implies

$$
\begin{aligned}
\tilde{F}_{-4}(r_B)r_B^{-\frac{c}{\epsilon}} - \tilde{F}_{-4}(r_A)r_A^{-\frac{c}{\epsilon}} &= \frac{c_{r2}}{\epsilon^4}\frac{\epsilon}{c}\Big(r_A^{-\frac{c}{\epsilon}} - r_B^{-\frac{c}{\epsilon}}\Big) + \int_{r_A}^{r_B} G_4(s)s^{-\frac{c}{\epsilon}}ds, \\
\tilde{F}_{-4}(r)r^{-\frac{c}{\epsilon}} &= \tilde{F}_{-4}(r_B)r_B^{-\frac{c}{\epsilon}} + \frac{c_{r2}}{\epsilon^4}\frac{\epsilon}{c}\Big(r_B^{-\frac{c}{\epsilon}} - r^{-\frac{c}{\epsilon}}\Big) \\
&\quad + \int_{r_B}^{r} dsG_4(s)s^{-\frac{c}{\epsilon}}.
\end{aligned}
$$

Eliminating c_{r2}, it follows that

$$\tilde{F}_{-4}(r) = \frac{(\frac{r}{r_A})^{\frac{c}{\epsilon}}-1}{(\frac{r_B}{r_A})^{\frac{c}{\epsilon}}-1}\tilde{F}_{-4}(r_B)+\frac{(\frac{r}{r_B})^{\frac{c}{\epsilon}}-1}{(\frac{r_A}{r_B})^{\frac{c}{\epsilon}}-1}\tilde{F}_{-4}(r_A)+\int_{r_A}^{r}\frac{(\frac{r_B}{s})^{\frac{c}{\epsilon}}-(\frac{r}{s})^{\frac{c}{\epsilon}}}{(\frac{r_B}{r_A})^{\frac{c}{\epsilon}}-1}G_4(s)ds$$
$$+\int_{r}^{r_B}\frac{(\frac{r}{s})^{\frac{c}{\epsilon}}-(\frac{r_A}{s})^{\frac{c}{\epsilon}}}{(\frac{r_A}{r_B})^{\frac{c}{\epsilon}}-1}G_4(s)ds. \tag{3.5}$$

The computation leading to (3.5) holds analogously for F and (3.1) with $\tilde{F}_4=(k_4F_4+F_{r^2\bar{A}})$. At this point we recall that m_4 has been defined by

$$m_4=k_4^{-1}\left(\frac{(\frac{r}{r_A})^{\frac{c}{\epsilon}}-1}{(\frac{r_B}{r_A})^{\frac{c}{\epsilon}}-1}\tilde{F}_4(r_B)+\frac{(\frac{r}{r_B})^{\frac{c}{\epsilon}}-1}{(\frac{r_A}{r_B})^{\frac{c}{\epsilon}}-1}\tilde{F}_4(r_A)\right). \tag{3.6}$$

Replace all moments of $F_{-\perp}$ of negative ϵ-order in G_4 with higher order ones, iteratively until all are of nonnegative order. E.g. $\int Mdv\tilde{J}(F_{-\perp},v_\theta)$ $v_r\bar{A}$ can be written as $\int MdvF_{\perp}\lambda$ and expressed by moments of higher order by projecting (3.2) along $\tilde{L}^{-1}\lambda$. This can be repeated until all appearing moments of $F_{-\perp}$ are of nonnegative order in ϵ. Notice that all appearing hydrodynamic moments are of ϵ-order zero or higher with a factor δ'. The negative ϵ-order F_r-moments were eliminated by the passage from F to F_-, and the integrals of the $\tilde{J}$-terms containing the remaining negative order hydrodynamic moments come out as zero, essentially because $\tilde{L}$ and $\tilde{L}^{-1}$ preserve even/odd symmetry under change of signs in v.

An analogous estimate for $\frac{\tilde{F}_{-\theta}}{r}:=\frac{w_1F_\theta}{r}+\frac{F_{-\theta r^2\bar{B}}}{r}$ can be obtained in the same way. Namely, multiply the Equation (3.2) with $Mv_rv_\theta\bar{B}(|v|)$ and integrate over $\mathbb{R}^3_v$. It follows that

$$\left(\frac{w_1F_\theta}{r}+\frac{F_{-\theta r^2\bar{B}}}{r}\right)'-\frac{F_{-\theta^3\bar{B}}-3F_{-\theta r^2\bar{B}}}{r^2} \tag{3.7}$$
$$=\frac{1}{r\epsilon^4}\left(\frac{c_{\theta r}}{r^2}+\frac{1}{r^2}\int_1^r s^2\frac{g_\theta}{\epsilon^4}+\int v_rv_\theta\bar{B}\tilde{J}\left(F_{-\perp},\sum_1^4\epsilon^j\varphi^j\right)Mdv\right)$$
$$+\eta_1\int_1^r s\frac{g_0}{\epsilon^3}ds+\epsilon^3F_\theta c_3k_6+\epsilon^4F_\theta\eta_2+\sum_{j=5}^{j_1}\epsilon^{j-4}\int v_rv_\theta\bar{B}\tilde{J}(F_-,\varphi^j)Mdv$$
$$+\frac{1}{\epsilon^4}\int v_rv_\theta\bar{B}Mgdv+\epsilon\int v_rv_\theta\bar{B}(F_r(1)\bar{\gamma}_1+m_\theta\bar{\gamma}_2+m_4\bar{\gamma}_3)Mdv.$$

Here, η_1 and η_2 are known coefficients. Making analogous computations to the $\tilde{F}_{-4}$-case leads to (3.3). ∎

Define a specular reflection operator $\mathcal{S}$ at $r=r_A,\ r_B$ as $\mathcal{S}f(r,v)=f(r,-v_r,v_\theta,v_z)$.

Proposition 3.1. Let F be a solution in $\mathcal{W}^{2-}$ to (3.1). The following estimate holds for small enough $\epsilon>0$;

$$|F_{\|}|_2 \leqslant c\Big(|F_{\perp}|_2+\frac{1}{\epsilon^8}|g_{\|}|_2+\frac{1}{\epsilon^4}|g_{\perp}|_2+|SF_-|_{\sim}+|F_b|_{\sim}\Big). \tag{3.8}$$

Proof of Proposition 3.1. Recall that the hydrodynamic moments of F and F_- coincide. Multiplying the Equation (3.1) with M and integrating over $\mathbb{R}^3_v$, leads to $(rF_r)'=r\frac{g_0}{\epsilon^4}$, i.e.

$$F_r(r)=\frac{F_r(1)}{r}+\frac{1}{r}\int_1^r s\frac{g_0}{\epsilon^4}ds. \tag{3.9}$$

By definition of $F_r(1)$,

$$\begin{aligned}|F_r(1)| &= \Big|\int v_rF_-(1,v)Mdv\Big|\\ &\leqslant c\left(\int |v_r|F_-^2(1,v)Mdv\right)^{\frac{1}{2}} \leqslant c(|SF_-|_{\sim}+|F_b|_{\sim}).\end{aligned}$$

It then follows from (3.9) that

$$\|F_r\|_2 \leqslant c\left(\frac{1}{\epsilon^4}|g_{\|}|_2+|SF_-|_2+|F_b|_{\sim}\right).$$

Multiply the Equation (3.2) with v_rM and integrate with respect to v. It follows that

$$\begin{aligned}\left(\int v_r^2F_-(r,v)Mdv\right)' &= \left(F_0+\sqrt{\frac{2}{3}}F_4+F_{-r^2}\right)'=\frac{F_{-\theta^2}-F_{-r^2}}{r}+\frac{g_r}{\epsilon^4}\\ &\quad +\epsilon\int v_r(F_r(1)\beta_1+m_\theta\beta_2+m_4\beta_3)Mdv.\end{aligned} \tag{3.10}$$

Multiply (3.10) with $2\Big(F_0+\sqrt{\frac{2}{3}}F_4+F_{-r^2}\Big)$ and integrate with respect to r on (r, r_B), then on (r_A, r_B), to obtain

$$\Big\|F_0+\sqrt{\frac{2}{3}}F_4\Big\|_2 \leqslant c\Big(|F_{-\perp}|_2 + \frac{1}{\epsilon^4}\|g_r\|_2 + \Big|\int v_r^2 F_-(r_B, v)M dv\Big| + \epsilon |F_r(1)| + |m_\theta|_2 + |m_4|_2\Big).$$

But

$$\Big|\int v_r^2 F_-(r_B, v)M dv\Big| \leqslant c\left(\int M|v_r|F_-^2(r_B, v)dv\right)^{\frac{1}{2}} \leqslant c(|SF_-|_\sim + |F_b|_\sim).$$

Hence

$$\Big\|F_0+\sqrt{\frac{2}{3}}F_4\Big\|_2 \leqslant c\Big(|F_{-\perp}|_2 + \frac{1}{\epsilon^4}\|g_r\|_2 + |SF_-|_\sim + |F_b|_\sim\Big). \quad (3.11)$$

By (3.3), (3.5), and (3.7)

$$\|F_4\|_2 + \|F_\theta|_2 \leqslant c\Big(|F_{-\perp}|_2 + \frac{1}{\epsilon^8}|g_\|\|_2 + \frac{1}{\epsilon^4}|g_\perp|_2 + |SF_-|_\sim + |F_b|_\sim + \epsilon\delta'|F_\|\|_2\Big).$$

And so, (3.8) holds. ∎

It remains to control $|SF_-|_\sim$ and the nonhydrodynamic part $F_\perp$.

Proposition 3.2. Let F be a solution in $\mathcal{W}^{\infty-}$ of (3.1) and F_- a solution in $\mathcal{W}^{2-}$ of (3.2) for $g=g_\perp$. The following estimates hold for small enough $\epsilon>0$;

$$\epsilon^2|\mathcal{S}F_-|_\sim + |\tilde{\nu}^{\frac{1}{2}}F_{-\perp}|_2 \leqslant c\Big(\epsilon^{-3}|\tilde{\nu}^{-\frac{1}{2}}g_\perp|_2 + \epsilon^{-7}|g_\||_2 + \epsilon^2\delta'(\|F_r\|_2 + \|F_\theta\|_2 + \|F_4\|_2 + \|F_0\|_2) + \epsilon^2|F_b|_\sim\Big), \quad (3.12)$$

$$|\tilde{\nu}^{\frac{1}{2}}F|_\infty \leqslant c(|\tilde{\nu}^{-\frac{1}{2}}g|_\infty + \epsilon^{-\frac{8}{q}}|\tilde{\nu}^{\frac{1}{2}}F|_q + |\tilde{\nu}^{\frac{1}{2}}F_b|_\sim), \quad q \leqslant \infty. \quad (3.13)$$

Proof of Proposition 3.2. We first turn to the estimate (3.13). Employing ref. 23, p. 101 for $\varphi = 0$, F can via a double iteration of the problem in exponential form, and splitting of the compact part K of $\tilde{L}$, be written as

$$F = U_\epsilon \frac{K'}{\epsilon^4} U_\epsilon \frac{K'}{\epsilon^4} F + Z_1 F + Z_2 g + Z_3 \gamma^+ F, \tag{3.14}$$

where

$$\left| \tilde{\nu}^{\frac{1}{2}} U_\epsilon \tfrac{K'}{\epsilon^4} U_\epsilon \tfrac{K'}{\epsilon^4} F \right|_\infty \leqslant c_\delta \epsilon^{-\frac{8}{q}} \,|\, \tilde{\nu}^{\frac{1}{2}} F \,|_q, \tag{3.15}$$
$$|\, \tilde{\nu}^{\frac{1}{2}} Z_1 F \,|_\infty \leqslant c\delta \,|\, \tilde{\nu}^{\frac{1}{2}} F \,|_\infty, \quad |\, \tilde{\nu}^{\frac{1}{2}} Z_2 g \,|_\infty \leqslant c \,|\, \tilde{\nu}^{-\frac{1}{2}} g \,|_\infty,$$
$$|\, \tilde{\nu}^{\frac{1}{2}} Z_3 \gamma^+ F \,|_\infty \leqslant c \,|\, \tilde{\nu}^{\frac{1}{2}} F_b \,|_\sim .$$

Using (3.14), (3.15) with δ a small enough constant, gives (3.13). For ϵ small enough, the addition of $\epsilon \tilde{J}(F, \varphi^1)$ to g does not change the result, nor does the addition of the higher order asymptotic terms. We notice that in this part of the proof, a hydrodynamic component in g does not change the proof.

For $\varphi = 0$ (3.1) coincides with (3.2). Then the mapping from $\tilde{\nu}^{-\frac{1}{2}} \tilde{L}^q \times L^+$ into $\mathcal{W}^{q-}$ given by $(g, F_b) \to F_-$, is continuous and bijective by (ref. 23, Ch. 6.1.). The analysis in ref. 23 is carried out for $2 \leqslant q \leqslant \infty$. Green's formula and the spectral inequality for $\tilde{L}$,

$$-\int M f \tilde{L} f \, dv \geqslant c \int M \tilde{\nu} f_\perp^2 \, dv$$

with $c > 0$, give

$$\epsilon^4 \,|\, \mathcal{S} F_- \,|_\sim^2 + |\, \tilde{\nu}^{\frac{1}{2}} F_{-\perp} \,|_2^2 \leqslant c \Big(\epsilon^4 \,|\, F_b \,|_\sim^2 + |\, \tilde{\nu}^{-\frac{1}{2}} g_\perp \,|_2^2 + \frac{1}{\epsilon^6} \,|\, g_\parallel \,|_2^2 + \epsilon^6 \,|\, F_\parallel \,|_2^2 \Big).$$

The case $\varphi \neq 0$ adds to the (g, F_-)-term

$$\int \left(\epsilon \tilde{J} \left(F_- - \frac{F_r(1)}{r} v_r - m_\theta v_\theta - m_4 \psi_4, \varphi^1 \right) + \sum_{j=2}^{4} \epsilon^j \tilde{J} \left(F_- - \frac{F_r(1)}{r} v_r, \varphi^j \right) \right)$$
$$\times F_- M \, dv \, dr.$$

There

$$\int \epsilon \tilde{J}\left(F_- - \frac{F_r(1)}{r}v_r - m_\theta v_\theta - m_4\psi_4, \varphi^1\right) F_- M dv dr$$
$$\leqslant \frac{\epsilon^2\delta'^2}{2\delta}\int \tilde{J}^2\left(F_- - \frac{F_r(1)}{r}v_r - m_\theta v_\theta - m_4\psi_4, v_\theta\right) M dv dr + \frac{\delta}{2}\mid F_{-\perp}\mid_2^2,$$

which is smaller than

$$\frac{c\delta'^2}{2\delta}\left(\epsilon^2 \parallel F_\perp \parallel_2^2 + \epsilon^4 \parallel F_\parallel \parallel_2^2 + \frac{1}{\epsilon^{14}}\mid g_\parallel \mid_2^2 + \frac{1}{\epsilon^6}\mid g_\perp \mid_2^2 + \epsilon^4 \mid \mathcal{S}F_- \mid_\sim^2 + \epsilon^4 \mid F_b \mid_\sim^2\right)$$
$$+ c\delta \mid F_{-\perp}\mid_2^2,$$

by the expressions of $F_- - \frac{F_r(1)}{r}v_r - m_\theta v_\theta - m_4\psi_4$ in terms of g_0, G_θ and G_4 given by (3.5), (3.7) and (3.9). It is here that the removal of $\frac{F_r(1)}{r}v_r + m_\theta v_\theta + m_4\psi_4$ from F_- in the ϵ^{-3}-term of Equation (3.2) satisfied by F_- plays a central role. The term

$$\int \sum_{j=2}^{4} \epsilon^j \tilde{J}\left(F_- - \frac{F_r(1)}{r}v_r, \varphi^j\right) F_- M dv dr$$

can be handled similarly. This completes the proof of (3.12). ∎

It directly follows from Proposition 3.1 and Proposition 3.2 that

Corollary 3.3. If $0 < \delta'$ is small enough and $g = g_\perp$, then for small enough $\epsilon > 0$ the following estimates hold for the solution of (3.1),

$$\mid \tilde{\nu}^{\frac{1}{2}} F_\perp \mid_2 \leqslant c\left(\epsilon^{-3}\mid \tilde{\nu}^{-\frac{1}{2}} g_\perp \mid_2 + \epsilon^2 \mid F_b \mid_\sim\right),$$
$$\parallel F_0 \parallel_2 + \parallel F_r \parallel_2 + \parallel F_4 \parallel_2 + \parallel F_\theta \parallel_2 \leqslant c\left(\epsilon^{-5}\mid \tilde{\nu}^{-\frac{1}{2}} g_\perp \mid_2 + \mid F_b \mid_\sim\right).$$

Using this corollary we prove

Proposition 3.4. Let $g = g_\perp$, $\tilde{\nu}^{-\frac{1}{2}} g \in \tilde{L}^q$, $F_b \in L^+$, $2 \leqslant q < \infty$, and $j_1 \geqslant 4$ be given. When $\delta' > 0$ is small enough, there exists a unique solution $F \in \mathcal{W}^{q-}$ to (3.1) for all small enough $\epsilon > 0$.

Proof of Proposition 3.4. By [ref. 23, pp. 68–69] there is a unique solution $F \in \mathcal{W}^{2-}$ for $\varphi = 0$. That still holds, if we add $\frac{1}{\epsilon^4}\tilde{J}(F,\varphi)$ to the right hand side of (3.1). Namely, for $\varphi = 0$ start from the integrated solution formula with a single iteration (cf ref. 23, p. 69),

$$F = U_\epsilon \frac{K}{\epsilon^4} F + U_\epsilon \frac{g}{\epsilon^4} + W_\epsilon \gamma^+ F. \tag{3.16}$$

Adding $\tilde{J}(F,\varphi)$ to g, a similar formula holds and, like (ref. 23 pp. 68–69), gives a compact perturbation of a well-posed problem. The index remains equal to zero, and the *a priori* estimates of Corollary 3.3 imply injectivity, hence also surjectivity. That completes the proof of the proposition in the $\mathcal{W}^{2-}$ case. From here the case $q = \infty$ follows using (3.13), and the case $2 < q < \infty$ from a corresponding generalization of (3.13). ∎

4. THE REST TERM

This section discusses the rest term, when $|u_{\theta A1}|$, $|u_{\theta B1}|$, and $|\tau_{B2}|$ are bounded by $\delta' > 0$, so that the results of the previous section hold. Given the asymptotic expansion φ, the aim is to prove that there exists a rest term R so that

$$f = M(1 + \varphi + \epsilon^4 R) \tag{4.1}$$

is an isolated solution to (1.1–2) with $M^{-1} f \in \tilde{L}^\infty$. This corresponds to the function R being a solution to

$$DR = \frac{1}{\epsilon^4}\left(\tilde{L}R + 2\tilde{J}(R,\varphi) + \epsilon^4 \tilde{J}(R,R) + l\right).$$

Notice that φ is constructed so that $D\varphi = (I - P_0)D\varphi$, hence that $l = l_\perp$. In Section 2 for the bifurcation case with $(u_{\theta A1} - u_{\theta B1} r_B)(3u_{\theta A1} + u_{\theta B1} r_B) > 0$, $\Delta \leqslant \Delta_{bif}$, an asymptotic expansion φ of order four in ϵ was constructed so that l is of ϵ-order one in $\tilde{L}^q$. Continue the same φ-expansion up to ϵ-order eighteen, giving an l-term of ϵ-order fifteen, but without requiring the boundary conditions to be satisfied for φ beyond order thirteen.

Let the sequences $(R^n)_{n \in I\!N}$ be defined by $R^0 = 0$, and

$$DR^{n+1} = \frac{1}{\epsilon^4}\left(\tilde{L}R^{n+1} + 2\sum_{j=1}^{18} \epsilon^j \tilde{J}(R^{n+1}, \varphi^j) + g^n\right), \tag{4.2}$$

$$R^{n+1}(1,v) = R_A(v), \quad v_r > 0, \qquad R^{n+1}(r_B, v) = R_B(v), \quad v_r < 0. \tag{4.3}$$

In (4.2–3)

$$g^n := \epsilon^4 \tilde{J}(R^n, R^n) + l,$$

$$\epsilon^4 R_A(v) := e^{\epsilon u_{\theta A1} v_\theta - \frac{\epsilon^2}{2} u_{\theta A1}^2} - 1 - \textstyle\sum_{j=1}^{18} \epsilon^j \varphi^j (r_A, v), \quad v_r > 0,$$

$$\epsilon^4 R_B(v) := \frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}} e^{\frac{1}{2}\left(v^2 - \frac{1}{1+\tau_B}(v_r^2 + (v_\theta - \epsilon u_{\theta B1})^2 + v_z^2)\right)} - 1 - \textstyle\sum_{j=1}^{18} \epsilon^j \varphi^j (r_B, v), \quad v_r < 0,$$

with R_A, R_B of ϵ-order ten.

We now turn to the properties of the rest term iteration scheme (4.2–3).

Proposition 4.1. For $\epsilon > 0$ and small enough, there is a unique sequence (R^n) of solutions to (4.2–3) in the set $X := \{R; |\tilde{\nu}^{\frac{1}{2}} R|_q \leqslant K\}$ for some constant K. The sequence converges in $\tilde{L}^q$ for $2 \leqslant q \leqslant \infty$, to an isolated solution of

$$DR = \frac{1}{\epsilon^4}\left(\tilde{L}R + \epsilon^4 \tilde{J}(R, R) + 2\tilde{J}(R, \varphi) + l\right), \tag{4.4}$$

$$R(1, v) = R_A(v), \quad v_r > 0, \qquad R(r_B, v) = R_B(v), \quad v_r < 0. \tag{4.5}$$

When ϵ tends to zero, the corresponding hydrodynamic moments of (4.1) converge to solutions of the (Hilbert) limiting fluid equations of leading order in ϵ (third order for the radial velocity).

Proof of Proposition 4.1. The existence result of Proposition 3.4 holds for the boundary value problem

$$Df = \frac{1}{\epsilon^4}\left(\tilde{L}f + 2\sum_{j=1}^{18} \epsilon^j \tilde{J}(f, \varphi^j) + g\right),$$

$$f(1, v) = R_A(v), \quad v_r > 0, \qquad f(r_B, v) = R_B(v), \quad v_r < 0.$$

Consider first (4.2–3) in the case $n = 0$ with $g^0 = l$. As discussed before (4.2), this $g^0 = g_\perp^0$ is of ϵ-order fifteen in $\tilde{L}^q$. For $R_\perp^1$ and $q \leqslant \infty$, Corollary 3.3 gives,

$$|\tilde{\nu}^{\frac{1}{2}} R_\perp^1|_2 \leqslant c\left(\epsilon^{-3} |\tilde{\nu}^{-\frac{1}{2}} g_\perp^0|_2 + \epsilon^2 |F_b|_\sim\right), \tag{4.6}$$

$$\| R_r^1 \|_2 + \| R_\theta^1 \|_2 + \| R_4^1 \|_2 + \| R_0^1 \|_2 \leqslant c\left(\epsilon^{-5} |\tilde{\nu}^{-\frac{1}{2}} g_\perp^0|_2 + |F_b|_\sim\right). \tag{4.7}$$

Using the properties of l, it follows from (4.6–7) that the ϵ-order of $R^1_\perp$ in $\tilde{L}^2$ is twelve, whereas the term $R^1_\parallel$ is of order ten in $\tilde{L}^2$. By Proposition 3.2, R^1 is of order six in $\tilde{L}^\infty$.

For $n \in \mathbb{N}$, we shall write $R^{n+1} = R^1 + \sum_{j=1}^{n}(R^{j+1} - R^j)$. It holds that $(R^{n+1} - R^n)$ has $g^n = g^n_\perp$ and the ingoing boundary values vanishing. Consider the case $n = 1$. By Corollary 3.3 for the difference $R^2 - R^1$,

$$| \tilde{\nu}^{\frac{1}{2}}(R^2_\perp - R^1_\perp) |_2 \leqslant c\epsilon \, | \tilde{\nu}^{-\frac{1}{2}} \tilde{J}(R^1, R^1) |_2, \tag{4.8}$$

$$\| R^2_r - R^1_r \|_2 + \| R^2_\theta - R^1_\theta \|_2 + \| R^2_4 - R^1_4 \|_2 + \| R^2_0 - R^1_0 \|_2 \leqslant c\epsilon^{-1} \, | \tilde{\nu}^{-\frac{1}{2}} \tilde{J}(R^1, R^1) |_2 . \tag{4.9}$$

Recall that

$$| \tilde{\nu}^{-\frac{1}{2}} \tilde{J}(g, h) |_q \leqslant C \, | \tilde{\nu}^{\frac{1}{2}} g |_\infty | \tilde{\nu}^{\frac{1}{2}} h |_q .$$

We conclude from this and from (4.8–9), that

$$| \tilde{\nu}^{\frac{1}{2}}(R^2 - R^1) |_q < c\epsilon \, | \tilde{\nu}^{\frac{1}{2}} R^1 |_q \quad \text{for } q = 2, \infty.$$

For $n \geqslant 2$, Corollary 3.3 implies that

$$| \tilde{\nu}^{\frac{1}{2}}(R^{n+1} - R^n) |_2 \leqslant \frac{c}{\epsilon} \, | \tilde{\nu}^{-\frac{1}{2}} (\tilde{J}(R^n, R^n) - \tilde{J}(R^{n-1}, R^{n-1})) |_2 .$$

And so (R^n) converges for sufficiently small $\epsilon > 0$ to some R, solution to (4.4–5) in $\tilde{L}^q$ for $q \leqslant \infty$. The contraction mapping construction guarantees that this solution is isolated.

It finally follows from the above proof that, when ϵ tends to zero, the hydrodynamic moments converge to the (Hilbert type) solutions of the corresponding leading order limiting fluid equations (2.14–15), (2.17). ∎

Proof for Theorem 1.2. This theorem is an immediate consequence of Proposition 2.1 and Proposition 4.1. ∎

The approach holds with small changes for the other cases of asymptotic expansion in the two-roll setup that are discussed in ref. 26. We let the case of Theorem 1.1 illustrate this.

Sketch of Proof for Theorem 1.1. Consider the boundary value problem (1.1–2), this time *without* the previous coupling (1.4) between the boundary values. Assume that the cylinders rotate in the same direction and that $1 < P_{SB2}/[(r_B^2 - 1)u_{\theta B1}^2] < (u_{\theta A1}/u_{\theta B1}r_B)^2$. This guarantees an asymptotic expansion with positive, as well as one with negative, second order radial velocity, and one with fourth order radial velocity (cf. ref. 25). Construct the expansions φ similarly to Section 2, and write the solution as in (4.1) with asymptotic expansion of order eighteen and rest term of order four. In the two cases of a second order radial velocity, the lowest nonvanishing v_r-term of φ appears in φ^2 and thereby gives a minor change in the proof of Section 3. In the case of a fourth order radial velocity, the lowest order nonvanishing v_r-term appears in φ^4, again giving a corresponding small change in the proof of Section 3. Except for this, the above proofs carry through as before. The rest term analysis also proceeds as before and proves Theorem 1.1. ∎

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